4.6 Montel's Theorem

Let X be a topological space. We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on X.

Definition 4.26. A topological space is called *separable* iff it contains a countable dense subset.

Definition 4.27. Let X be a topological space, $F \subseteq \mathcal{C}(X)$. F is called *pointwise bounded* iff for each $a \in X$ there is a constant $M > 0$ such that $|f(a)| < M$ for all $f \in F$. F is called *locally bounded* iff for each $a \in X$ there is a constant $M > 0$ and a neighborhood $U \subseteq X$ of a such that $|f(x)| < M$ for all $x \in U$ and for all $f \in F$.

Definition 4.28. Let X be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *equicontinuous at* $a \in X$ iff for every $\epsilon > 0$ there exists a neighborhood $U \subseteq X$ of a such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all} \quad x, y \in U.$$

A subset $F \subseteq \mathcal{C}(X)$ is called *locally equicontinuous* iff F is equicontinuous at a for all $a \in X$.

Definition 4.29. Let X be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *normal* iff every sequence of elements of F has a subsequence that converges uniformly on every compact subset of X.

Theorem 4.30 (Arzela-Ascoli)**.** *Let X be a separable topological space and $F \subseteq \mathcal{C}(X)$. Suppose that F is pointwise bounded and locally equicontinuous. Then, F is normal.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of F. We have to show that there exists a subsequence that converges uniformly on any compact subset of X. We encode subsequences of a sequence through infinite subsets of $\mathbb{N}$ in the obvious way. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points which is dense in X. Set $N_0 := \mathbb{N}$ and construct iteratively $N_k \subseteq N_{k-1}$ as follows. The sequence $\{f_n(x_k)\}_{n\in N_{k-1}}$ is bounded by the assumption of pointwise boundedness of F. Thus there exists a convergent subsequence given by an infinite subset $N_k \subseteq N_{k-1}$. Proceeding in this way we obtain a sequence of decreasing infinite subsets $N_0 \supset N_1 \supset N_2 \supset \dots$. Now consider the sequence $\{n_l\}_{l\in\mathbb{N}}$ of strictly increasing natural numbers n_l obtained as follows: n_l is the lth element of the set N_l. It is then clear that the sequence $\{f_{n_l}(x_k)\}_{l\in\mathbb{N}}$ converges for every $k \in \mathbb{N}$.

Now let $K \subseteq X$ be compact and choose $\epsilon > 0$. Since F is locally equicontinuous, we find for each $a \in K$ an open neighborhood $U_a \subseteq X$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ if $x, y \in U_a$. Since K is compact there are finitely many points $a_1, \dots, a_m \in K$ such that $U_{a_1}, \dots, U_{a_m}$ cover K. Since $\{x_k\}_{k\in\mathbb{N}}$ is dense in X there exists for each $j \in 1, \dots, m$ an index k_j such that $x_{k_j} \in U_{a_j}$. Now, $\{f_{n_l}(x_{k_j})\}_{l\in\mathbb{N}}$ converges and is Cauchy for all $j \in \{1, \dots, m\}$. In particular, by taking a maximum if necessary we can find $l_0 \in \mathbb{N}$ such that $|f_{n_i}(x_{k_j}) - f_{n_l}(x_{k_j})| < \epsilon$ for all $i, l \geq l_0$ and for all $j \in \{1, \dots, m\}$.

Now fix $p \in K$. Then, there is $j \in \{1, \dots, m\}$ such that $p \in U_{a_j}$. For $i, l \geq l_0$ we thus obtain the estimate

$$\begin{aligned} |f_{n_i}(p) - f_{n_l}(p)| \leq{}& |f_{n_i}(p) - f_{n_i}(x_{k_j})| \\ &+ |f_{n_i}(x_{k_j}) - f_{n_l}(x_{k_j})| + |f_{n_l}(x_{k_j}) - f_{n_l}(p)| < 3\epsilon. \end{aligned}$$

In particular, this implies that $\{f_{n_l}\}_{l\in\mathbb{N}}$ converges uniformly on K. □

Theorem 4.31 (Montel)**.** *Let $D \subseteq \mathbb{C}$ be a region and $F \subseteq \mathcal{O}(D)$. Suppose that F is locally bounded. Then, F is normal.*

Proof. We show that F is locally equicontinuous. The result follows then from the Arzela-Ascoli Theorem 4.30. Let $z_0 \in D$ and choose $\epsilon > 0$. Since F is locally bounded, there exists a constant $M > 0$ and $r > 0$ with $\overline{B_{2r}(z_0)} \subset D$ and such that $|f(z)| < M$ for all $z \in \overline{B_{2r}(z_0)}$ and all $f \in F$. The Cauchy Integral Formula (Theorem 2.20) yields for all $f \in F$ and $z, w \in B_{2r}(z_0)$

$$\begin{aligned} f(z) - f(w) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{2r}(z_0)} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) \mathrm{d}\zeta \\ &= \frac{z - w}{2\pi \mathrm{i}} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} \mathrm{d}\zeta. \end{aligned}$$

If we restrict to $z, w \in B_r(z_0)$ we have the estimate $|(\zeta - z)(\zeta - w)| > r^2$ for all $\zeta \in \partial B_{2r}(z_0)$. Combining this with the standard integral estimate (Proposition 2.7) we obtain,

$$|f(z) - f(w)| \leq |z - w| \frac{2\|f\|_{\partial B_{2r}(z_0)}}{r} < |z - w| \frac{2M}{r}.$$

Choosing $\delta := \min\left\{r, \frac{r\epsilon}{4M}\right\}$ yields the estimate

$$|f(z) - f(w)| < \epsilon \quad \forall z, w \in B_\delta(z_0),$$

showing local equicontinuity. This completes the proof. □

Exercise 58. Let X be a metric space and $F \subseteq \mathcal{C}(X)$. Suppose that F is normal. Show that F is locally bounded.

Exercise 59 (Vitali's Theorem)**.** Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a locally bounded sequence of holomorphic functions on D. Let $f \in \mathcal{O}(D)$ and $A := \{z \in D : \lim_{n\to\infty} f_n(z) \text{ exists and } f(z) = \lim_{n\to\infty} f_n(z)\}$. Suppose that A has a limit point in D. Show that $f_n \to f$ uniformly on compact subsets of D for $n \to \infty$.

4.7 The Riemann Mapping Theorem

Proposition 4.32. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of holomorphic functions $f_n \in \mathcal{O}(D)$ that converges uniformly on any compact subset of D to f. Then, $f \in \mathcal{O}(D)$ and the sequence $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges uniformly on any compact subset of D to $f^{(k)}$ for all $k \in \mathbb{N}$.*

Proof. Let $z_0 \in D$ and set $r > 0$ such that $\overline{B_r(z)} \subset D$. By Corollary 2.15 f_n is integrable in $B_r(z_0)$. For any closed path γ in $B_r(z_0)$ we thus have

$$\int_\gamma f = \int_\gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\gamma f_n = 0,$$

where we have used Proposition 2.8 to interchange the integral with the limit. Thus, f is integrable in $B_r(z_0)$ and hence holomorphic there by Corollary 2.23. Since the choice of z_0 was arbitrary we find that f is holomorphic in all of D.

Fix $k \in \mathbb{N}$ and consider $z_0 \in D$. Choose $r > 0$ such that $\overline{B_{2r}(z_0)} \subseteq D$. Now for each $z \in B_r(z_0)$ we have the Cauchy estimate (Proposition 2.31),

$$|f_n^{(k)}(z) - f^{(k)}(z)| \le \frac{k!}{r^k} \|f_n - f\|_{\partial B_r(z)} \le \frac{k!}{r^k} \|f_n - f\|_{\overline{B_{2r}(z_0)}}.$$

For $\epsilon > 0$ there is by uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ an $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon\, r^k/k!$ for all $n \ge n_0$ and all $z \in \overline{B_{2r}(z_0)}$. Hence, $|f_n^{(k)}(z) - f^{(k)}(z)| < \epsilon$ for all $n \ge n_0$ and all $z \in B_r(z_0)$. That is, $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges to $f^{(k)}$ uniformly on some neighborhood of every point of D. To obtain uniform convergence on a compact subset $K \subset D$ it is merely necessary to cover K with finitely many such neighborhoods. □

Theorem 4.33 (Hurwitz)**.** *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Let $a \in D$ and $r > 0$ such that $\overline{B_r(a)} \subset D$. Suppose that $f(z) \neq 0$ for all $z \in \partial B_r(a)$. Then, there exists $n_0 \in \mathbb{N}$ such that f and f_n have the same number of zeros in $B_r(a)$ for all $n \ge n_0$.*

Proof. Set $\delta := \inf\{|f(z)| : z \in \partial B_r(a)\}$. By the assumptions $\delta > 0$ and $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\partial B_r(a)$. Thus, there exists $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \delta/2$ for all $n \geq n_0$ and all $z \in \partial B_r(a)$. But this implies,

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)| \quad \forall n \geq n_0, \forall z \in \partial B_r(a).$$

Applying Rouché's Theorem 3.21 yields the desired result. □

Proposition 4.34. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Suppose that for all $n \in \mathbb{N}$, f_n has no zeros. Then, either $f = 0$ or f has no zeros.*

Proof. **Exercise.** □

Proposition 4.35. *Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of injective functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of D to f. Then, either f is constant or f is injective.*

Proof. Suppose that f is not constant. Let a in D and set $p := f(a)$ and $p_n := f_n(a)$ for all $n \in \mathbb{N}$. By injectivity $f_n - p_n$ never vanishes on $D \setminus \{a\}$. On the other hand, the sequence $\{f_n - p_n\}_{n\in\mathbb{N}}$ converges uniformly in any compact subset of D to $f - p$. Since $f - p \neq 0$, Proposition 4.34 implies that $f - p$ has no zeros in $D \setminus \{a\}$. In other words, f does not take the value p at any point of $D \setminus \{a\}$. Since we chose a arbitrarily it follows that f is injective. □

Theorem 4.36 (Riemann Mapping Theorem). *Every homologically simply connected region which is different from $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$.*

Proof. Let D be the region in question. Fix $z_0 \in D$ arbitrarily. Let $F \subseteq \mathcal{O}(D)$ be the set of holomorphic functions $f \in \mathcal{O}(D)$ which are injective, whose image is contained in $\mathbb{D}$ and such that $f(z_0) = 0$. Our strategy is to find an element of F which is a biholomorphism $D \to \mathbb{D}$.

First we show that F is not empty. By assumption $D \neq \mathbb{C}$, so we can choose $a \in \mathbb{C} \setminus D$. The function $f(z) := z - a$ is holomorphic and zero-free in D, so according to Theorem 4.25 there is a holomorphic square root $g \in \mathcal{O}(D)$ with $g^2 = f$. If $g(z_1) = g(z_2)$ then $(g(z_1))^2 = (g(z_2))^2$ and so $z_1 = z_2$ since f is injective. Therefore also g is injective. Moreover, if $g(z_1) = -g(z_2)$ we can draw the same conclusion $z_1 = z_2$, but this time we get a contradiction, since g is zero-free. Thus, if $z \in \mathbb{C}$ is in the image of g, then $-z$ cannot be in the image of g. Now since g is not constant the

Open Mapping Theorem 2.40 ensures that $g(D)$ is open. In particular there exists $w \in \mathbb{C}$ and $r > 0$ such that $\overline{B_r(w)} \subset g(D)$. But applying the previous statement to all elements of $B_r(w)$ we obtain $\overline{B_r(-w)} \cap g(D) = \emptyset$. It is now easy to see that the function $h \in \mathcal{O}(D)$ defined by $h(z) := r/(g(z) + w)$ is also injective and satisfies $h(D) \subseteq \mathbb{D}$. Setting $v := h(z_0)$, we have $D_v \circ h \in F$ since $D_v \in \mathrm{Aut}(\mathbb{D})$ and $D_v(v) = 0$.

Since D is open, there exists $r > 0$ such that $\overline{B_r(z_0)} \subset D$. Using the Cauchy estimate (Proposition 2.31) we find the bound $|f'(z_0)| < 1/r$ for all $f \in F$. This implies that

$$M := \sup\{|f'(z_0)| : f \in F\}$$

is well defined. On the other hand we will show that if $f(D) \neq \mathbb{D}$ for some $f \in F$, then there exists $g \in F$ such that $|g'(z_0)| > |f'(z_0)|$. This implies that $h \in F$ is a biholomorphism $D \to \mathbb{D}$ iff $|h'(z_0)| = M$. We will then show that such an h exists.

Consider some $f \in F$ such that $f(D) \neq \mathbb{D}$. Choose $p \in \mathbb{D} \setminus f(D)$. Since $D_p \in \mathrm{Aut}(\mathbb{D})$, the composition $D_p \circ f$ is injective and $D_p \circ f(D) \subset \mathbb{D}$. Furthermore, $D_p \circ f$ is zero-free since $D_p^{-1}(0) = \{p\}$. Since D is homologically simply connected we can find a holomorphic square root $g \in \mathcal{O}(D)$ with $g^2 = D_p \circ f$ according to Theorem 4.25. In fact, it is clear that g is injective and $g(D) \subseteq \mathbb{D}$. Set $w := g(z_0)$. Then $h := D_w \circ g \in F$. Consider now the holomorphic map $k : \mathbb{D} \to \mathbb{D}$ given by $k(z) = D_p((D_w(z))^2)$. Then, $f = k \circ h$ and applying the chain rule for derivatives we obtain

$$f'(z_0) = k'(h(z_0))h'(z_0) = k'(0)h'(z_0).$$

Noting that $k(0) = 0$ we can apply the Schwarz Lemma 4.11. Since k is not a rotation, this implies $|k'(0)| < 1$. Hence, $|f'(z_0)| < |h'(z_0)|$ since $h'(z_0) \neq 0$ by injectivity of h.

The image of all functions in F is contained in the bounded set $\mathbb{D}$, so in particular F is locally bounded. According to Montel's Theorem 4.31 this implies that F is normal. Consider now a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of F such that $|f_n'(z_0)| \to M$ as $n \to \infty$. Since F is normal, there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges uniformly on any compact subset of D to a function $f \in \mathcal{O}(D)$ by Proposition 4.32. By the same Proposition we have convergence of the derivative and thus $|f'(z_0)| = M$ as desired. It remains to show that $f \in F$. From the limit process it is clear that $f(z_0) = 0$ and $f(D) \subseteq \overline{\mathbb{D}}$. Since f is not constant (in particular, $f'(z_0) \neq 0$) the Open Mapping Theorem 2.40 implies that $f(D)$ must be open and so we must have $f(D) \subseteq \mathbb{D}$. The injectivity of f follows from Proposition 4.35. Hence $f \in F$. This completes the proof. $\square$

Proposition 4.37. *Let $D \subset \mathbb{C}$ be a homologically simply connected region, $a \in D$. Then, there exists exactly one biholomorphism $f : D \to \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

Proof. **Exercise.** $\square$

Exercise 60. Show that a homologically simply connected region cannot be conformally equivalent to a region that is not homologically simply connected.